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Geometric contacts of surfaces immersed in $\mathbb{R}^n$, $n \geqslant 5$

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ABSTRACT

We study the extrinsic geometry of surfaces immersed in $\mathbb{R}^n$, $n \geqslant 5$, by analyzing their contacts with different standard geometrical models, such as hyperplanes and hyperspheres. We investigate the relation between different types of contact and the properties of the curvature ellipses at each point. In particular, we focalize our attention on the hyperspheres having contacts of corank two with the surface. This leads in a natural way to the concept of umbilical focus and umbilic curvature.

1. Introduction

The analysis of the contacts of a submanifolds with special submanifolds (models) which are invariant for a given group acting on the ambient space lies in the basis of the description of their geometrical properties. Following a suggestions of R. Thom, I.R. Porteous introduced in [21] useful tools, based on the study of the generic singularities of the family of distance-squared functions on submanifolds of $\mathbb{R}^n$, in order to provide an accurate description of their contacts with hyperspheres. Since then, several authors have produced a good number of works in which these kind of techniques are applied to the investigation of special problems of local and global type concerning the geometry submanifolds in Euclidean, Affine and Minkowski spaces (see, for instance [1,4,6–9,12–14,22,24]).

The geometrical properties attached to the contacts of the surfaces in $\mathbb{R}^5$ with hyperplanes were analyzed in [14] in terms of the generic singularities of the family of height functions. In the present work we study the local geometry of surfaces immersed in $\mathbb{R}^n$, $n \geqslant 5$, in a wider context that includes the local approximation by hyperspheres. For this purpose, we consider the singularities of distance squared functions and their relation with the behavior of the curvature ellipses at different points of the surfaces. The concept of curvature ellipse of a surface immersed in Euclidean space was first introduced in [18] and the particular case of surfaces immersed in $\mathbb{R}^4$ was treated with full details in [10].

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Particularly interesting points in this context are the corank two singularities of the distance squared functions on the surfaces. For surfaces in $\mathbb{R}^3$, these points turn out to be the well known umbilics, i.e. the critical points of their principal configurations. On the other hand, J. Montaldi characterized in [16] the corank 2 singularities of distance squared functions on surfaces in $\mathbb{R}^4$ as the points at which the curvature ellipse was degenerated and called them semiumbilics. The semiumbilics form, generically, smoothly embedded curves in these surfaces and it has been shown more recently [19] that they can also be characterized as critical points of principal configurations associated to the different normal fields on the surface and thus have a special meaning from the dynamical viewpoint. It follows from the application of standard techniques in the study of singularities of generic families of functions, that for surfaces immersed in higher codimension, the corank two singularities of the distance squared functions may occur over open subsets of the surface. The main problem studied in this paper concerns the generic distribution of these points and of their corresponding focal centers (umbilical foci) in the focal set of the surface.

In Section 2 we introduce the notation and basic properties regarding the curvature ellipses and their relations with some geometrical aspects of surfaces immersed in $\mathbb{R}^n$, $n \geqslant 4$. Section 3 is devoted to the study of contacts of surfaces with hyperplanes in connection with the properties of the curvature ellipse at each point. We give a description of the generic contacts of the surface with hyperplanes in $\mathbb{R}^n$, $n \geqslant 4$, which extends some the results for surfaces in $\mathbb{R}^4$ and $\mathbb{R}^5$ respectively obtained in [13] and [14]. We characterize here the cone of degenerate normal directions (Proposition 3.5) and the existence of contacts of corank 2 (referred to the corank of the corresponding height function) of the surface with hyperplanes in terms of the curvature ellipse at each point. Section 4 contains a characterization of the special contacts of a surface M with hyperspheres in $\mathbb{R}^n$, $n \geqslant 5$, in terms of the different singularities of the family of distance-squared functions on M. As a consequence, we obtain the *characterization of the semiumbilic points as singularities of corank 2 for distance-squared functions* (Theorem 4.2) which generalizes Montaldi's result [16]. This leads in a natural way to the definitions of umbilical foci and umbilical focal subset. This subset is generically an algebraic variety of codimension 2 in the focal set of the surface M in $\mathbb{R}^n$. An important feature consists in the fact that *there is at least one umbilical focus at each point of the surface, except at the radial semiumbilics and flat umbilics, for which the umbilical focal subset lies at infinity*. From all possible focal centers at each point there is a unique one lying in the first normal space of the surface at a given point. Theorem 4.2 provides the expression of such a focal center in terms of the distance of the subspace spanned by the curvature ellipse to the considered point. This setting allows us to introduce in Section 5 the concept of *umbilical curvature* at a point p of a surface in $\mathbb{R}^n, n \geqslant 5$, as the distance of p to the affine plane $\mathcal{A}ff_p$ determined by its curvature ellipse in N_pM. We study some of its properties and show that for surfaces contained in hyperspheres this distance is constant. We remark that this is not a sufficient condition for hypersphericity. In fact, we observe that surfaces lying in the envelope of a two-parameter family of hyperspheres of fixed radii also have constant umbilical curvature. We also consider the problem of isometric reduction of the codimension in terms of properties of the curvature ellipses and the umbilical curvature. As a consequence, we see that *a simply connected surface M immersed in $\mathbb{R}^n$, $n \geqslant 5$, with non-vanishing constant umbilical curvature, all whose points are semiumbilic, can be isometrically immersed in 3-sphere. In case of vanishing umbilical curvature, they can be isometrically immersed in a 3-dimensional affine subspace*.

Finally, we include in the last section a table whose entries provide a summary of the relations between the type of the curvature ellipses and the existence of osculating hyperplanes of corank 2 and focal hyperspheres at each point.

2. The second fundamental form and curvature ellipses

Let M be a surface immersed in $\mathbb{R}^n$, $n \geqslant 4$, locally described as image of a C^2-embedding $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^n$. For each point $p \in M$ consider the decomposition $T_p\mathbb{R}^n = T_pM \oplus N_pM$, where N_pM denotes the normal space to M at p. Let $\bar{\nabla}$ denote the Riemannian connection of $\mathbb{R}^n$. Given vector fields, X, Y, locally defined along M, we can consider local extensions $\bar{X}, \bar{Y}$ to open sets of $\mathbb{R}^n$, and define the Riemannian connection on M as $\nabla_X Y = (\bar{\nabla}_{\bar{X}} \bar{Y})^{\top}$, the tangential component of the connection $\bar{\nabla}$.

If we denote respectively by $\mathcal{X}(M)$ and $\mathcal{N}(M)$ the spaces of tangent and normal fields on M, the second fundamental form on M is the normal component of $\bar{\nabla}_{\bar{X}}\bar{Y}$ and is given by:

$$\alpha_\phi : \mathcal{X}(M) \times \mathcal{X}(M) \longrightarrow \mathcal{N}(M),$$
$$(X, Y) \longmapsto \alpha_\phi(X, Y) = \bar{\nabla}_{\bar{X}}\bar{Y} - \nabla_X Y = d^2\phi(X, Y)^{\perp},$$

this is a well defined bilinear symmetric map.

Now, for each $p \in M$ and $\nu \in N_pM$, $\nu \neq \vec{0}$, this induces a quadratic form

$$II_\nu : T_pM \longrightarrow \mathbb{R},$$
$$v \longmapsto II_\nu(v) = \nu \cdot \alpha_\phi(v, v) = \nu \cdot d^2\phi(v, v)^{\perp}.$$

Let $\phi : U \subset \mathbb{R}^2 \to \mathbb{R}^n$ be a local parametrization of M, given by $\phi(x, y)$, the functions $E, F, G : U \to \mathbb{R}$ defined by $E = \phi_x(x, y) \cdot \phi_x(x, y)$, $F(x, y) = \phi_x(x, y) \cdot \phi_y(x, y)$ and $G(x, y) = \phi_y(x, y) \cdot \phi_y(x, y)$ are called coefficients of the first fundamental form on M. We can take $w_1 = \frac{\phi_x}{E}$ and $w_2 = \frac{E\phi_y - F\phi_x}{\sqrt{E(EG-F^2)}}$ vector fields in $\mathcal{X}(M)$ that define an orthonormal tangent frame. Then the coefficients of the second fundamental form for ν are:

$$e_\nu = \alpha(w_1, w_1)\cdot\nu = \frac{1}{E}\phi_{xx}\cdot\nu,$$
$$f_\nu = \alpha(w_1, w_2)\cdot\nu = \frac{1}{E\sqrt{EG-F^2}}(E\phi_{xy} - F\phi_{xx})\cdot\nu,$$
$$g_\nu = \alpha(w_2, w_2)\cdot\nu = \frac{1}{E(EG-F^2)}\big(E^2\phi_{yy} - 2EF\phi_{xy} + F^2\phi_{xx}\big)\cdot\nu.$$

If we take local coordinates $\{x, y\}$ and an orthonormal frame, $\{e_1, e_2, e_3, \ldots, e_n\}$, in a neighbourhood of $p = \phi(0,0) \in M$, such that $\{e_1, e_2\}$ is a tangent frame and $\{e_3, \ldots, e_n\}$ is a normal frame, the matrix of the second fundamental form is given by

$$\alpha_\phi(p) = \begin{bmatrix} a_1 & b_1 & c_1 \\ & \vdots & \\ a_{n-2} & b_{n-2} & c_{n-2} \end{bmatrix},$$

where

$$a_i = \frac{\partial^2\phi}{\partial x^2}(0,0)\cdot e_{i+2}, \qquad b_i = \frac{\partial^2\phi}{\partial x\partial y}(0,0)\cdot e_{i+2}, \qquad c_i = \frac{\partial^2\phi}{\partial y^2}(0,0)\cdot e_{i+2}, \tag{1}$$

$i = 1, \ldots, n-2$, are the coefficients of the second fundamental form for the normal frame.

Given $p \in M$, consider the unit circle in T_pM parameterized by the angle $\theta \in [0, 2\pi]$ with respect to the direction e_1. Denote by γ_θ the curve obtained by intersecting M with the hyperplane at p composed by the direct sum of the normal subspace N_pM and the straight line in the tangent direction represented by θ. Such curve is called normal section of $\phi(M)$ in the direction θ. The curvature vector $\eta(\theta)$ of γ_θ in p lies in N_pM. Varying θ from 0 to 2π, this vector describes an ellipse in N_pM, called the **curvature ellipse** of M at p. In fact, the curvature ellipse is the image of the affine map

$$\eta : S^1 \subset T_pM \longrightarrow N_pM$$

given by

$$\theta \longmapsto \eta(\theta) = \sum_{i=1}^{n-2}[\cos\theta \;\; \sin\theta].\begin{bmatrix} a_i & b_i \\ b_i & c_i \end{bmatrix}.\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\cdot e_{i+2},$$

that is,

$$\eta(\theta) = H + \cos(2\theta)B + \sin(2\theta)C,$$
$$\text{where} \quad H = \frac{1}{2}\sum_{i=1}^{n-2}(a_i + c_i)e_{i+2}, \quad B = \frac{1}{2}\sum_{i=1}^{n-2}(a_i - c_i)e_{i+2} \quad \text{and} \quad C = \sum_{i=1}^{n-2} b_i e_{i+2}. \tag{2}$$

The normal field H evaluated at a point $p \in M$ is known as the *mean curvature* vector of M at p.

The **first normal space** of M at p, denoted by N_p^1M, is the subspace of N_pM generated by $\{\phi_{xx}(p), \phi_{xy}(p), \phi_{yy}(p)\}$. It is easy to see that

$$N_p^1M = \big\langle \phi_{xx}(p), \phi_{xy}(p), \phi_{yy}(p)\big\rangle = \langle B, C, H\rangle.$$

We also denote by E_p the linear subspace of N_p^1M which is parallel to the affine hull $\mathcal{A}ff_p$ of the curvature ellipse. That is, $E_p = \langle B,\ C\rangle$. The orthogonal complements of E_p and N_p^1M in N_pM are respectively denoted by $E_p^\perp$ and $(N_p^1M)^\perp$.

In what follows we shall consider $N_pM = N_p^1M \times (N_p^1M)^\perp$.

The curvature ellipse $\eta_p(\theta)$ may degenerate into a segment at certain points $p \in M$. Following the nomenclature introduced in [15] for the case of surfaces in $\mathbb{R}^4$, the point p shall be called **semiumbilic**. Moreover, $\eta(\theta)$ may become a radial segment at certain points that we shall call **radial semiumbilics**, or even shrunk into a point, in which case p will said to be an **umbilic**. When the curvature ellipse coincides with the point p itself, we shall say that p is a **flat umbilic**.

We can also classify points of M according the following criterion:

$$\mathbf{p} \in \mathbf{M_i} \text{ or } \mathbf{p\ is\ of\ type\ M_i} \quad \text{if and only if} \quad \dim N_p^1M = i.$$

Now, by taking into account the dimension of E_p, the points of type M_i can be subdivided into the following subclasses:

- p is of type $M_3 \Leftrightarrow \{\phi_{xx}(0,0), \phi_{xy}(0,0), \phi_{yy}(0,0)\}$ are linearly independent, i.e., ϕ is non-singular of second order at p in Feldman's [3] sense $\Leftrightarrow$ the curvature ellipse is non-degenerated at p and $\mathcal{A}ff_p$ does not contain the origin.
- If a point p of type M_2, the curvature ellipse is either non-degenerate and $\mathcal{A}ff_p$ goes through the origin, if $\dim E_p = 2$, or a non-radial segment, if $\dim E_p = 1$. In this last case p is a non-radial semiumbilic point.

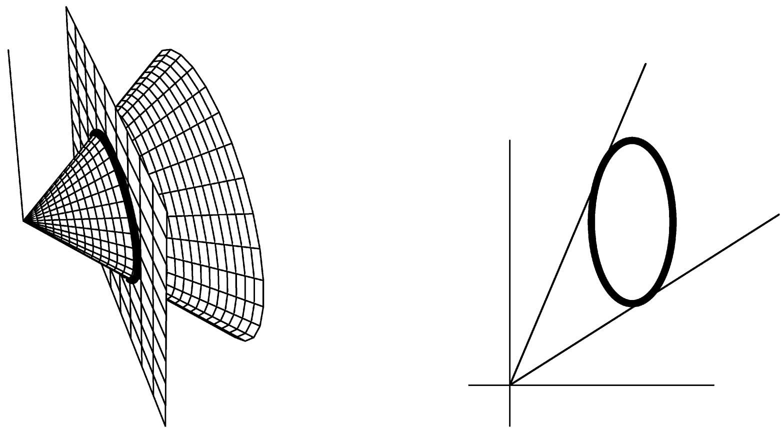

Fig. 1. The curvature ellipse and the cone $\mathcal{K}_p$ associate to the osculating paraboloid at a point p: On left p is a point of type M_3; On the right p is a non-semiumbilic point of type M_2.

- If p is of type M_1, it will be either a radial semiumbilic point, if $\dim E_p = 1$ (the curvature ellipse is a radial segment) or a non-flat umbilic, if $\dim E_p = 0$.
- p is of type $M_0 \Leftrightarrow p$ is flat umbilic $\Leftrightarrow N_p^1 M = \{0\}$.

We now introduce the concept of osculating paraboloid at a point p of a surface M immersed in $\mathbb{R}^n$, $n \geqslant 5$. This is done in terms of the curvature ellipse.

Suppose that M is locally given at $p \equiv \phi(0,0)$ by an immersion, in Monge form as

$$\begin{aligned}\phi : \big(\mathbb{R}^2, (0,0)\big) &\longrightarrow \big(\mathbb{R}^n, (0,\dots,0)\big),\\ (x,y) &\longmapsto xe_1 + ye_2 + \phi_1(x,y)e_3 + \cdots + \phi_{n-2}(x,y)e_n,\end{aligned}$$

where $\{e_1, e_2, e_3, \dots, e_n\}$ is an orthonormal frame as in the previous section. We shall say that the map

$$\begin{aligned}\mathbb{PO}_p : \mathbb{R}^2 &\longrightarrow \mathbb{R}^n,\\ (x,y) &\longmapsto xe_1 + ye_2 + p_1(x,y)e_3 + \cdots + p_{n-2}(x,y)e_n,\end{aligned}$$

given by the second order Taylor expansion of ϕ at p, describes the **osculating paraboloid** of M in p, where $p_i(x,y) = \frac{1}{2}(a_i x^2 + 2b_i xy + c_i y^2)$ and a_i, b_i and c_i are the coefficients of the second fundamental form defined above in (1).

It follows from the above definition that the osculating paraboloid of M at a point p is characterized by having the same tangent plane and curvature ellipse at p.

The osculating paraboloid of an immersion ϕ given in the Monge form can also be written as

$$\mathbb{PO}_p(x,y) = xe_1 + ye_2 + x^2\frac{\phi_{xx}(0,0)}{2} + xy\phi_{xy}(0,0) + y^2\frac{\phi_{yy}(0,0)}{2}.$$

The image of its quadratic part will be contained in the subspace generated by the vectors $\phi_{xx}(0,0)$, $\phi_{xy}(0,0)$ and $\phi_{yy}(0,0)$. If these vectors are linearly independent, the curvature ellipse at p in this 3-dimensional space will be precisely the intersection of the double-cone given by this quadratic part,

$$\mathcal{K}_p(u,v) = u^2\frac{\phi_{xx}(0,0)}{2} + uv\phi_{xy}(0,0) + v^2\frac{\phi_{yy}(0,0)}{2},$$

with the plane $\gamma(s,t) = H + tB + sC$, where $u = \cos\theta$, $v = \sin\theta$, $t = \cos 2\theta$ and $s = \sin 2\theta$ (Fig. 1).

We observe that the quadratic part of the osculating paraboloid determines a cone spanned by the curvature ellipse vectors with vertex at the origin.

In what follows we will consider an orthonormal frame $\{e_1, e_2, e_3, \dots, e_n\}$ such that $E_p \subset \langle e_3, e_4\rangle$ and $N_p^1 M \subset \langle e_3, e_4, e_5\rangle$, so the vectors of $N_p^1 M$ described in (2) are written as $H = \frac{1}{2}((a_1 + c_1)e_3 + (a_2 + c_2)e_4 + (a_3 + c_3)e_5)$, $B = \frac{1}{2}((a_1 - c_1)e_3 + (a_2 - c_2)e_4)$ and $C = b_1 e_3 + b_2 e_4$.

3. Cone of degenerate directions: Contacts with hyperplanes

The description of a submanifold contacts with hyperplanes and hyperspheres can be characterize [16] of by means of the analysis of the singularities of height and squared-distance functions respectively. We apply this method in order to describe all possible contacts of a surface with the hyperplanes and hyperspheres of $\mathbb{R}^n$.

The **family of height functions** associated to an immersion $\phi : \mathbb{R}^2 \to \mathbb{R}^n$ of a surface M in $\mathbb{R}^n$, $n \geqslant 4$, is given by

$$\Phi : \mathbb{R}^2 \times S^{n-1} \longrightarrow \mathbb{R},$$
$$(x, v) \longmapsto \Phi(x, v) = \phi_v(x) = \phi(x) \cdot v.$$

We observe that given $v \in S^{n-1}$, a point $p \in M$ is a singular point of ϕ_v if and only if $v \in N_pM$.

It follows from genericity theorems ([11] and [17]) in singularity theory that there exists a residual subset of immersions $\mathcal{I}$ in $C^\infty(M, \mathbb{R}^n)$ with the Whitney C^∞ topology, such that the height functions family associated to each $\phi \in \mathcal{I}$ is locally stable [11,17]. In such case, the germ of the height function in most normal directions $v \in S^{n-1}$ at a point p is of Morse type, but may degenerate for some of these directions at each point $p \in M$ [12]. The normal directions leading to a degenerate germ of height function at a given point $p \in M$ shall be called **degenerate normal directions** of M at the point p. A degenerate **normal direction** at p will be said to be of **corank** r provided the corresponding height function has a singularity of corank r at p. Hyperplanes normal to these directions will have higher contact with the surface and shall be called **higher contact hyperplanes**.

For a generic surface in $\mathbb{R}^5$, the degenerate singularities of most height functions are of type A_2 (folds) and it is possible find at least one and at most five normal directions for which p is a singularity of cusp type or worse (A_k, $k \geqslant 3$) at each point p of type M_3 [14]. Such directions are called binormals and their associated contact hyperplanes are the osculating hyperplanes of the surface at the considered point. The corresponding contact directions are known as asymptotic directions. The generic behavior of their integral curves are described in [23].

Proposition 3.1. *(See [20].) Given a submanifold M immersed in $\mathbb{R}^n$, $n \geqslant 4$, a point $p \in M$ and a non-null vector $\upsilon \in N_pM$, the quadratic forms $II_v(p)$ and the Hessian for the height function associated to υ, $\mathrm{Hess}(\phi_v(p))$, are equivalent.*

In the case of an immersion ϕ of a surface M in $\mathbb{R}^n$ given in Monge form and with an orthonormal frame $\{e_1, \dots, e_n\}$ defined in a neighbourhood of $p = \phi(0,0)$, $\{e_1, e_2\} \subset T_pM$, this result follows immediately from the fact that given $v \in N_pM$, the v-second fundamental form $II_v(p)$ and $\mathrm{Hess}(\phi_v(p))$ have the same matrix:

$$II_v(p) = \begin{bmatrix} \phi_{xx}(0,0) \cdot v & \phi_{xy}(0,0) \cdot v \\ \phi_{xy}(0,0) \cdot v & \phi_{yy}(0,0) \cdot v \end{bmatrix} = \mathrm{Hess}\big(\phi_v(p)\big).$$

The height function ϕ_v associated to a normal vector, $v \in N_pM$ has a *non-Morse* or a *degenerate singularity* at p if only if the determinant of $\mathrm{Hess}(\phi_v)(p)$ vanishes. We say that p is a singularity of corank r of some function ϕ on M provided the corank of the quadratic form $\mathrm{Hess}(\phi)(p)$ is r. If ϕ_v has corank r $(r = 1, 2)$ at p, we shall say that v is a degenerate normal direction of corank r and the corresponding higher contact hyperplane shall also be said to be corank r.

Definition 3.2. Given a surface M immersed in $\mathbb{R}^n$ and a point $p \in M$, we define the **cone of degenerate directions** at p as the set of vectors v in N_p^1M such that ϕ_v has a degenerate singularity at p. We denote it by $\mathcal{C}_p$ (see Fig. 2).

Proposition 3.3 *(Cone of degenerate directions). Given an immersion $\phi : \mathbb{R}^2 \to \mathbb{R}^n$, $n \geqslant 4$, and $v \in N_pM$, we can assert*:

(i) *The height function $\phi_v : \mathbb{R}^2 \to \mathbb{R}$, associated to a normal vector, $v \in N_pM$ has a non-Morse or a degenerate singularity at p if only if $(H \cdot v)^2 = (B \cdot v)^2 + (C \cdot v)^2$.*
(ii) *p is a singularity of corank 2 for ϕ_v if only if v is in the orthogonal complement of the first normal space $v \in (N_p^1M)^\perp \subset N_pM$.*
(iii) *Condition (i) is equivalent to $v \in \mathcal{C}_p \times (N_p^1M)^\perp$ and if we consider the immersion ϕ in the Monge form with the special frame that contains the curvature ellipse principal axes, the set $\mathcal{C}_p \subset N_p^1M$ of the degenerate directions (of corank 1) can be expressed as*

$$\mathcal{C}_p = \bigg\{xe_3 + ye_4 + ze_5 \in N_p^1M;\ \frac{1}{4}\big((a_1 + c_1)x + (a_2 + c_2)y + (a_3 + c_3)z\big)^2 = \frac{1}{4}\big((a_1 - c_1)x + (a_2 - c_2)y\big)^2 + (b_1x + b_2y)^2\bigg\}. \tag{3}$$

Proof. The assertions of this proposition are consequences of the proposition above and the replacements:

$$\phi_{xx}(0,0) = H + B, \quad \phi_{yy}(0,0) = H - B \quad \text{and} \quad \phi_{xy}(0,0) = C. \qquad \square$$

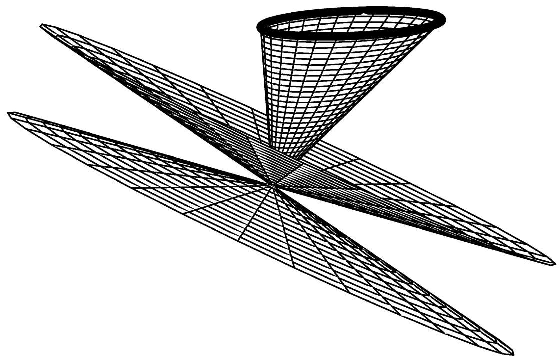

Fig. 2. N_p^1M for $\phi(x,y)=(x,y,x^2,2xy,x^2-y^2)$ at $p=\phi(0,0)$: The curvature ellipse, the cone $\mathcal{K}_p$ given by the osculating paraboloid and the double cone $\mathcal{C}_p$ of the degenerate directions.

Remark 3.4.

1. Given a surface M immersed in $\mathbb{R}^n$, $n \geqslant 5$, we see from (ii) in the above proposition that all the normal directions leading to height functions with a corank 1 singularity at p define a $(n-5+i)$-dimensional subspace of N_pM if only if $p \in M_i$.
2. For surfaces immersed in $\mathbb{R}^4$, the expression (3) in the above proposition can be replaced by $\mathcal{C}_p = \{xe_3 + ye_4 \in N_p^1M;\ \frac{1}{4}((a_1+c_1)x+(a_2+c_2)y)^2 = \frac{1}{4}((a_1-c_1)x+(a_2-c_2)y)^2 + (b_1x+b_2y)^2\}$, the conic of degenerate normal directions. Note also that for those surfaces, p is a corank 2 singularity of some height function on M if and only if it is either an umbilic or a radial semiumbilic point, since we must have $\dim(N_p^1M) < 2$. This result was shown in [12].
3. As a straightforward consequence of item (ii) of the last proposition we get the following results which were proven in [13] for surfaces in $\mathbb{R}^5$: (a) Points of type M_3 are singularities of corank 1 for each one of the height functions associated to the degenerate directions. (b) A point $p \in M$ is of type M_2 or M_1 according to there is a unique direction or a plane of directions (a circle of normal directions) whose associated height function has a corank 2 singularity at p.

Next proposition describes the cone $\mathcal{C}_p \subset N_p^1M$ of degenerate directions of an immersion at a point p relating it to the curvature ellipse at this point.

Proposition 3.5. *Given a surface M immersed in $\mathbb{R}^n$, $n \geqslant 5$ and $p \in M$, we have*:

1. *$p \in M_3 \Leftrightarrow \mathcal{C}_p$ is a non-degenerate double-cone.*
2. *Points of type M_2 can be subdivided as*:
   (a) *Either is p a non-radial semiumbilic point or the origin is outside the curvature ellipse at $p \Leftrightarrow \mathcal{C}_p$ is a pair of crossing lines in the N_p^1M-plane. These lines are perpendicular to the boarding lines of the cone $\mathcal{K}_p$ determined by the curvature ellipse.*
   (b) *The origin belongs to the curvature ellipse at $p \Leftrightarrow \mathcal{C}_p$ is a line in the E_p-plane. This line is perpendicular to the border line of the cone spanned by the curvature ellipse.*
   (c) *The origin is inside the curvature ellipse in N_p^1M then $\mathcal{C}_p = \{0\}$ (all degenerate directions are of corank 2 and lie on the $(n-4)$-dimensional space $(N_p^1M)^\perp = E_p^\perp$).*
3. *Points of type M_1 can be subdivided as*:
   (a) *The origin is an end point of the segment defined by the degenerate curvature ellipse $\Rightarrow \mathcal{C}_p = N_p^1M$-line.*
   (b) *Either p is umbilic, or the origin is not an end point of the segment defined by the degenerate curvature ellipse $\Rightarrow \mathcal{C}_p = \{0\}$ (all degenerate directions are of corank 2 and lie on the $(n-3)$-dimensional space $(N_p^1M)^\perp$).*
4. *$p \in M_0 \Rightarrow \mathcal{C}_p = \{0\}$ (all degenerate directions are of corank 2 and lie on the $(n-2)$-dimensional space $(N_p^1M)^\perp$).*

Proof. 1. We consider the non-degenerate double-cone $\widetilde{\mathcal{C}}_p = \{xe_3 + ye_4 + ze_5;\ \frac{1}{4}((a_1+c_1)x+(a_2+c_2)y+(a_3+c_3)z)^2 = \frac{1}{4}((a_1-c_1)x+(a_2-c_2)y)^2 + (b_1x+b_2y)^2\}$ in $\langle e_3,e_4,e_5\rangle$. Since $p \in M_3 \Leftrightarrow N_p^1M = \langle e_3,e_4,e_5\rangle$, then we have that $p \in M_3 \Leftrightarrow \mathcal{C}_p = \widetilde{\mathcal{C}}_p$.

2. Since $p \in M_2 \Leftrightarrow \dim N_p^1M = 2$ we can consider $N_p^1M = \langle e_3,e_4\rangle$ and in this case the quadric that defines $\mathcal{C}_p$ is given by the equation

$$Q(x,y):\ \left(a_1c_1-b_1^2\right)x^2 + (a_1c_2+a_2c_1-2b_1b_2)xy + \left(a_2c_2-b_2^2\right)y^2 = 0.$$

The discriminant of Q is given by $\Delta(p) = (a_1c_1-b_1^2)(a_2c_2-b_2^2) - \frac{1}{4}(a_1c_2+a_2c_1-2b_1b_2)$ and of classification of quadrics we know that if $\Delta(p) > 0$ the quadric is a degenerate ellipse and there are no solutions; if $\Delta(p) = 0$ the quadric is a degenerate parabola and the solution is a line passing by the origin and if $\Delta(p) < 0$ the quadric is a degenerate hyperbola and the solution is a pair of lines crossing in the origin (see Fig. 3).

On the other side, it follows a result of [12] that if $\Delta(p) > 0$ the point p lies inside the curvature ellipse; if $\Delta(p) = 0$ the point p lies on the curvature ellipse and if $\Delta(p) < 0$ the point p lies outside the curvature ellipse.

3. Now, since $p \in M_1 \Leftrightarrow \dim N_p^1 M = 1$ we can consider $N_p^1 M = \langle e_3 \rangle$ and in this case the quadric that define $\mathcal{C}_p$ is given by $(a_1c_1 - b_1^2)x^2 = 0$, whose solution is a line if $a_1c_1 - b_1^2 = 0$ or a point if $a_1c_1 - b_1^2 \neq 0$, but for [12] we have that in this case $k(p) = a_1c_1 - b_1^2$ is the Gaussian curvature of M in p. Moreover, if $k(p) = 0$ then p is an end point of the segment ellipse and if $k(p) \neq 0$ then p is not an end point of the segment or p is umbilic point.

4. This case is immediate, since $\dim N_p^1 M = 0$ and all degenerate directions of $N_p M = (N_p^1 M)^\perp$ are of corank 2. □

We conclude this section with the following result which will be needed also in the next section.

Lemma 3.6. *For an immersion ϕ of a surface M in $\mathbb{R}^n$, $n \geqslant 5$, expressed in a frame $\{e_1, e_2, e_3, \ldots, e_n\}$ such that $\{e_1, e_2\} \subset TpM$, $p \in M$, $v \in N_p M$, we have*:

$$\operatorname{Hess}\big(\phi_v(p)\big) = \begin{bmatrix} \lambda_p(v) & 0 \\ 0 & \lambda_p(v) \end{bmatrix}, \quad \textit{if only if} \quad v \in E_p^\perp \ \big(\lambda_p(v) = v \cdot H\big).$$

Proof. The proof is direct just remarking that $v \in E_p^\perp \Leftrightarrow v \cdot B = v \cdot C = 0 \Leftrightarrow v \cdot \phi_{xx}(0,0) = v \cdot \phi_{yy}(0,0) = v \cdot H$ and $v \cdot \phi_{xy}(0,0) = 0$. □

Remark 3.7. As a first consequence of the above lemma we can assert that the set $E_p^\perp \cap H^\perp$ of all degenerate directions lying in $E_p^\perp$ are associate to a corank 2 singularity at p of the height functions.

4. Focal points: Contacts with hyperspheres

In this section we study the second order contacts of surfaces immersed in $\mathbb{R}^n$, $n \geqslant 5$, with hyperspheres. This is done through the analysis of the singularities of the family of distance-squared functions over the surfaces. This family is defined as follows, suppose that $M \subset \mathbb{R}^n$, $n \geqslant 4$, is locally given in the Monge form by an immersion $\phi : \mathbb{R}^2 \to \mathbb{R}^n$, then the distance-squared function from a point $a \in \mathbb{R}^n$ is given by

$$\begin{aligned} &d_a : \mathbb{R}^2 \to \mathbb{R}, \\ &(x, y) \mapsto d_a(x, y) = \big\| a - \phi(x, y) \big\|^2. \end{aligned}$$

Varying the point a in $\mathbb{R}^n$ we obtain a family of functions on M that describes all the possible contacts of M with the hyperspheres on $\mathbb{R}^n$. We observe that the function d_a has a singularity in a point $p = \phi(0,0) \in M$ if and only if $\phi_x(0,0) \cdot (a - p) = \phi_y(0,0) \cdot (a - p) = 0$ which is equivalent to ask that the point a lies in the normal subspace of M at p $(v = (a - p) \in N_p M)$.

It also follows from genericity theorems ([11] and [17]) that there is for a residual subset of immersions in $C^\infty(M, \mathbb{R}^n)$ most distance-squared functions associated to such an immersion are stable. Nevertheless, the study of the structure of the subset of points leading to non-stable distance-squared functions is very relevant from the geometrical viewpoint. In this sense, we have the following:

Definition 4.1. Given a surface M immersed in $\mathbb{R}^n$, $n \geqslant 5$, we say that the point $a \in \mathbb{R}^n$ is a **focal center** at $p \in M$ provided the distance-squared function d_a has a degenerate singularity at p. We denote by $\mathcal{F}_p$ the **set of focal centers associated** to p.

The subset of $\mathbb{R}^n$ made of all the focal centers for all the points of M is an algebraic variety that for a generic immersion has dimension $n - 1$. We shall refer to it here as the **focal hypersurface** of M. On the other hand, we can also consider a focal center at p as a point in $N_p^1 M$. The collection of all the focal centers of M, considered as a subset of the normal bundle NM, shall be called the **focal set** of M.

Any hypersphere tangent to M at p and centered at a focal center of M at p is said to be a **focal hypersphere** of M at p.

The study of the focal set from this viewpoint was introduced by I.R. Porteous in [21]. It was there shown that it coincides with the singular set of the normal exponential map $\exp_M : NM \to \mathbb{R}^n$. The properties of this set for surfaces immersed in $\mathbb{R}^3$ are extensively described in [21]. The case of surfaces in $\mathbb{R}^4$ was treated in [15].

We observe that the contacts of M with the focal hyperspheres take place along the directions contained in the kernel of the quadratic form $\operatorname{Hess}(d_a)$.

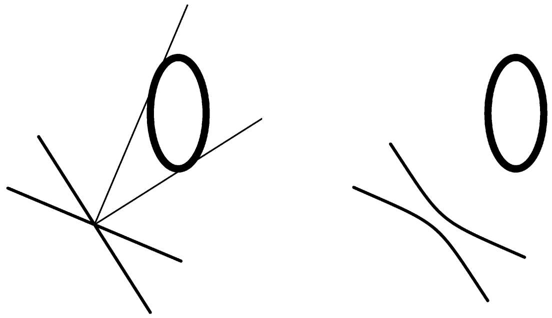

Fig. 3. At a non-semiumbilic point p of type M_2: On the left its curvature ellipse, the "cone" $\mathcal{K}_p$ and the "cone" $\mathcal{C}_p$ of degenerate directions in $N_p^1 M$; On the right its curvature ellipse and focal set in $N_p^1 M$.

Assuming the immersion ϕ in Monge form, we have, for $v = a - p \in N_p M$, that

$$\operatorname{Hess}\bigl(d_a(0,0)\bigr) = -2\begin{bmatrix} \phi_{xx}(0,0)\cdot v - 1 & \phi_{xy}(0,0)\cdot v \\ \phi_{xy}(0,0)\cdot v & \phi_{yy}(0,0)\cdot v - 1 \end{bmatrix}.$$

Therefore, the focal set of this immersion at $p = \phi(0,0)$ can be described as $\mathcal{F}_p = p + \mathcal{W}_p$, where

$$\mathcal{W}_p = \bigl\{v \in N_p M;\ \bigl(\phi_{xx}(0,0)\cdot v - 1\bigr)\bigl(\phi_{yy}(0,0)\cdot v - 1\bigr) - \bigl(\phi_{xy}(0,0)\cdot v\bigr)^2 = 0\bigr\}$$

and also can be written as

$$\mathcal{W}_p = \bigl\{v \in N_p M;\ (H\cdot v - 1)^2 = (B\cdot v)^2 + (C\cdot v)^2\bigr\}.$$

We note, from this last expression that if $v \in (N_p^1 M)^\perp \subset N_p M$ and denoting $v_{N_p^1 M}$ the orthogonal projection of v in $N_p^1 M$, then the focal set can be given as

$$\mathcal{F}_p = p + \bigl\{v \in N_p M;\ (H\cdot v_{N_p^1 M} - 1)^2 = (B\cdot v_{N_p^1 M})^2 + (C\cdot v_{N_p^1 M})^2\bigr\}.$$

We observe that the quadratic parts of $\mathcal{W}_p$ and $\mathcal{C}_p$ coincide. Now, by considering the immersion in the Monge form and taking a frame $\{e_1, e_2, e_3, \dots, e_n\}$ such that $N_p^1 M \subset \langle e_3, e_4, e_5\rangle$ as above, by using similar arguments to those used in Proposition 3.5 for the discussion of the characteristic polynomial associated to the quadratic part of the equation which defines $\mathcal{F}_p$, we get the next theorem concerning the geometric description of the set of focal centers at a given point. We also remark that it follows from the above expression for $\operatorname{Hess}(d_a(0,0))$ that the corank 2 condition for a focal center $p = a + v$ is equivalent to the second fundamental form being the identity. By relating this condition to Lemma 3.6, we can distinguish the corank two points on the whole set of focal points. Summing up we get:

Theorem 4.2. *Given a surface M immersed in $\mathbb{R}^n$, $n \geqslant 5$, $p \in M$ and consider $H_{E_p^{\perp *}}$, the projection of the mean curvature vector $H(p)$ on $E_p^{\perp *} = E_p^\perp \cap N_p^1 M$. The set of focal centers associated to p can be given as $\mathcal{F}_p = p + \mathcal{W}_p$, where*:

1. *If $p \in M_3 \Rightarrow \mathcal{W}_p = \{$a cone with vertex at $\frac{H_{E_p^{\perp *}}}{H_{E_p^{\perp *}}\cdot H_{E_p^{\perp *}}}\} \times (N_p^1 M)^\perp$ and the corank 2 focal points at p are*

$$\left\{\frac{H_{E_p^{\perp *}}}{H_{E_p^{\perp *}}\cdot H_{E_p^{\perp *}}}\right\} \times \bigl(N_p^1 M\bigr)^\perp.$$

(*We provide in Fig. 4 an example of the focal set and the curvature ellipse at a point of type M_3.*)

2. *For $p \in M_2$ and $\dim(E_p) = 2$ all focal points at p have corank 1 and we may have*:
   (a) *If p is inside the curvature ellipse then $\mathcal{W}_p = \emptyset$.*
   (b) *If p is outside the curvature ellipse then*

$$\mathcal{W}_p = \{\text{a hyperbolae in } E_p\} \times \bigl(N_p^1 M\bigr)^\perp.$$

   (c) *If p is on the curvature ellipse we have*

$$\mathcal{W}_p = \{\text{a parabola in } E_p\} \times \bigl(N_p^1 M\bigr)^\perp.$$

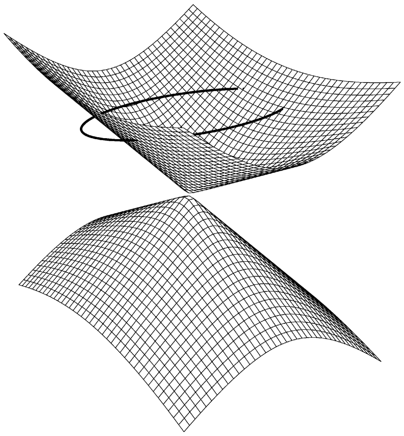

Fig. 4. The curvature ellipse and the focal set in $N_p^1 M$ at $p = \phi(0,0)$, for $\phi(x,y) = (x, y, x^2 - y^2, xy, x^2 + y^2)$.

3. *For $p \in M_2$ and p semiumbilic* $(\dim(E_p) = 1)$, *we have*
$$\mathcal{W}_p = \left\{ \textit{a pair of lines crossing at } \frac{H_{E_p^{\perp *}}}{H_{E_p^{\perp *}} \cdot H_{E_p^{\perp *}}} \right\} \times \left(N_p^1 M\right)^{\perp},$$
and the corank 2 *focal points are*
$$\left\{ \frac{H_{E_p^{\perp *}}}{H_{E_p^{\perp *}} \cdot H_{E_p^{\perp *}}} \right\} \times \left(N_p^1 M\right)^{\perp}.$$
4. *If $p \in M_1$ and p umbilic $\Rightarrow \mathcal{W}_p = \{\frac{H}{H \cdot H}\} \times (N_p^1 M)^{\perp}$ and affine subspace of dimension* $(n-3)$ *and all focal points at p have corank* 2.
5. *If $p \in M_1$ and p a radial semiumbilic then*
$$\mathcal{W}_p = \{H + B\} \times \left(N_p^1 M\right)^{\perp} \cup \{H - B\} \times \left(N_p^1 M\right)^{\perp}$$
and all focal points at p have corank 1.
6. *If $p \in M_0 \Rightarrow \mathcal{W}_p = \emptyset$.*

Among all the focal hyperspheres, there are some special ones corresponding to distance-squared functions with corank 2 singularities. The remaining part of this section shall be devoted to characterize them.

Definition 4.3. Given a surface M immersed in $\mathbb{R}^n$ and a point $p \in M$, a focal center of M at p is said to be an **umbilical focus** of M at p if and only if the corresponding distance-squared function has a singularity of corank 2 at p. The corresponding focal hypersphere shall be called **umbilical focal hypersphere at** p.

The case of surfaces immersed in $\mathbb{R}^4$ was analyzed in [15], where it was shown that a point $p \in M$ is a (non-radial) semiumbilic if and only if it is a contact point of M with some focal hypersphere centered at some umbilical focus at p. The radial semiumbilics are known as inflection points in this case. At such points, the umbilic center goes to infinity and the focal hypersphere becomes a osculating hyperplane [12]. The characterization of the umbilical foci can be given as a straightforward consequence of Theorem 4.2:

Corollary 4.4. *Given a surface M immersed in $\mathbb{R}^n$, $n \geqslant 4$, a point $p \in M$ and a non-null vector $v \in N_p M$ we have that $a = p + v$ is an umbilic focus of M at p, if only if, $H_{E_p^{\perp *}}$, the projection of the mean curvature vector on $E_p^{\perp *} = E_p^{\perp} \cap N_p^1 M$, is non-vanishing and $v = \frac{H_{E_p^{\perp *}}}{H_{E_p^{\perp *}} \cdot H_{E_p^{\perp *}}} + w$, where $w \in (N_p^1 M)^{\perp} \subset N_p M$.*

Corollary 4.5. *Let M be a surface immersed in $\mathbb{R}^n$, $n \geqslant 4$, and $p \in M$, then*:

1. *There is an umbilical focus at $p \in M$ (and a second order contact with a hypersphere) if only if $p \in M_3$ or p is non-radial semiumbilic or p is non-planar umbilic.*
2. *In all this cases, there is a unique umbilical focus a_p belonging to $N_p^1 M$, $a_p = p + \frac{H_{E_p^{\perp *}}}{H_{E_p^{\perp *}} \cdot H_{E_p^{\perp *}}}$ and the focal hypersphere at p of minimum radius is the one centered at a_p and its radius is $r_p = \frac{1}{d(p, \mathcal{A}ff_p)}$.*

Remark 4.6. If condition 1. is not verified and $n \geqslant 5$, we may say that the focal center at p in $N_p^1 M$, goes to infinity, since $E_p^{\perp} = (N_p^1 M)^{\perp} \subset N_p M \neq \emptyset$ and hence it follows from Proposition 3.5 that there is a second order contact with a hyperplane.

Corollary 4.7. *Given a surface M immersed in $\mathbb{R}^n$, $n \geqslant 5$, we have*

(i) *If $p \in M_3$, then there is $(n-5)$-affine subspace contained in $E_p^{\perp}$ made of umbilical foci for M at p.*
(ii) *If $p \in M_2$ is a semiumbilic point, then the umbilical foci of M at p fill a $(n-4)$-affine subspace contained in $E_p^{\perp}$.*

Remark 4.8. In the case that M is a surface immersed in $\mathbb{R}^5$, for $p \in M$ we have that: a) If p is a point of type M_3 then M has a unique umbilical focus. b) If p is a non-radial semiumbilic point then M has a line of umbilical foci.

5. Umbilical curvature

From Corollary 4.5 in the previous section the following definition arises quite naturally:

Definition 5.1. Let M be a surface immersed in $\mathbb{R}^n, n \geqslant 5$, which is not contained in any 4-dimensional subspace and $p \in M$. We define the **umbilical curvature** of M at p as $\kappa_u(p) = d(p, \mathcal{A}ff_p) = \|H_{E_p^{\perp *}}\|$.

Remark 5.2. It is interesting to note that for a surface immersed in $\mathbb{R}^5$ with non-degenerate curvature ellipses we may also consider an umbilical curvature with sign. Under the natural identification of $N_p M$ with $\mathbb{R}^3$ it can be defined as $\kappa_u(p) = H \cdot \frac{B \times C}{\|B \times C\|}$.

The following properties of the umbilical curvature function can be easily deduced from Lemma 3.6, Theorem 4.2 and Corollary 4.5:

a) If $M \subset \mathbb{R}^5$ and $p \in M_3$, then $\kappa_u(p)$ is the curvature of the hypersphere tangent to M centered at the unique umbilical focus at p.
b) If $M \subset \mathbb{R}^n, n > 5$, and $p \in M_3$, $\kappa_u(p)$ is the maximum curvature among those of all the tangent hyperspheres centered at umbilical foci of M at p.
Since $M \subset \mathbb{R}^n, n \geqslant 5$, we have:
c) If $p \in M_2$ is a non-semiumbilic point, then $\kappa_u(p) = 0$.
d) If $p \in M_2$ is a semiumbilic point, $\kappa_u(p)$ is the maximum curvature among those of all the tangent hyperspheres centered at umbilical foci of M at p.
e) If $p \in M_1$ is a radial semiumbilic point, then $\kappa_u(p) = 0$.
f) If $p \in M_1$ is an umbilic point, then $\kappa_u(p)$ is the maximum curvature among those of all the tangent hyperspheres centered at umbilical foci of M at p.
g) If $p \in M_1$ is a flat umbilic, then $\kappa_u(p) = 0$.

Remark 5.3. The umbilical curvature is a continuous function on the complementary of the subset of semiumbilic and umbilic points. It can be seen (as a consequence of the Thom's Transversality Theorem [5]) that this subset is made of isolated points on generically immersed surfaces in $\mathbb{R}^5$ and that generically immersed surfaces in $\mathbb{R}^n$, $n > 5$, do not have semiumbilic points. In the case of a surface M contained in $\mathbb{R}^4$, the plane defined by the curvature ellipse at a non-semiumbilic point p coincides with the normal plane of M at p. In this sense, we can extend the concept of umbilic curvature to these surfaces by saying that it vanishes at all their non-semiumbilic points. In fact, we can view M as a submanifold of a hyperplane H in $\mathbb{R}^5$. This hyperplane can be considered as a degenerate umbilical focal hypersphere at every point of M. We also observe that the semiumbilic points form closed regular curves on generically immersed surfaces in $\mathbb{R}^4$ [15], and we have that at a semiumbilic (resp. at an umbilic) point, the curvature ellipse defines an affine line (resp. a point), whose distance to the origin coincides with the curvature of the unique umbilical focal 3-sphere at the point (resp. of that with minimal radius). This 3-sphere becomes a 3-plane at radial semiumbilic (resp. flat umbilic) points, for which the umbilic curvature vanishes. By taking this into account we can also define the umbilical curvature on the semiumbilical surfaces of $\mathbb{R}^4$ in an analogous way to the general case of surfaces in $\mathbb{R}^n$, $n \geqslant 5$.

Given a normal field ν on M, we can consider its associated shape operator

$$S_\nu : T_p M \to T_p M, \quad S_\nu(X) = -(\bar{\nabla}_{\bar{X}} \bar{\nu})^{\top},$$

where $\bar{\nu}$ is a local extension of ν to $\mathbb{R}^5$ and $(-)^{\top}$ means the tangential component. Clearly, the second fundamental form can be expressed by $II_\nu(X) = S_\nu(X) \cdot X$, so the matrix of the operator S_ν at any point $p \in M$ coincides with that of the quadratic form $II_{\nu(p)}$. Then, given any $p \in M$, there exists an orthonormal basis of eigenvectors of $S_\nu \in T_p M$, for which the restriction of the second fundamental form to the unitary vectors, $II_\nu|_{S^1}$, takes its maximal and minimal values. The corresponding eigenvalues k_1, k_2 are the **maximal** and **minimal** **ν-principal curvatures**, respectively. The point p is a **ν-umbilic** if the ν-principal curvatures coincide. It follows from [19] that p is a ν-umbilic if and only if $\nu \in E_p^{\perp *}$. The field ν is said to be **umbilical** on M provided p is a ν-umbilic, $\forall p \in M$ ($\nu(p) \in E_p^{\perp *}$, $\forall p \in M$). In this case, the value of $II_\nu(\theta)$, $\theta \in S^1 \subset T_pM$ does not depend on the choice of the direction θ, but only of the point p. We denote it by κ_ν and call it the **curvature associated to the umbilical field** ν.

Remark 5.4. Given a normal field ν on M, it follows from Proposition 3.1 that the matrix of the operator S_ν at any point $p \in M$ coincides with the quadratic form $\mathrm{Hess}(\phi_\nu(p))$. Then Lemma 3.6 implies that ν is an umbilical normal field on M if and only if $\nu \in E_p^\perp$. In this case, provided ν is a unit field, $\kappa_\nu(p) = \nu \cdot H_{E_p^{\perp *}}$.

We remember that a normal vector field ν is **parallel** if $d^2\phi(X(p), \nu) = 0$ for any $X_p \in T_pM$ and for any $p \in M$, where $d^2\phi(X(p), \nu)$ is the normal component of $\bar{\nabla}_{\bar{X}}\bar{\nu}$. A normal sub-bundle is said to be **parallel** if it admits some local frame of parallel fields at every point.

In the next we give a characterization of the hypersphericity of surfaces in $\mathbb{R}^5$ in terms of the properties of their curvature ellipses.

Theorem 5.5. *Given a surface M immersed in $\mathbb{R}^n$, $n \geqslant 5$, all whose points are type M_3. Then M lies in a 4-sphere if and only if E is a parallel sub-bundle, where E is the sub-bundle defined by the planes $\{E_p\}$ of the curvature ellipses. Moreover, in this case the curvature κ_u is constant all over M and coincides with the curvature of the 4-sphere.*

Proof. Consider a unit normal field $\nu(p) \in E_p^\perp \cap N_p^1M$, $\forall p \in M$. Since $E_p^\perp \cap N_p^1M$ has dimension 1, we can take ν pointing towards $\mathcal{A}ff_p$ all over M. It follows from Corollary 4.4 that ν is umbilic over M. Since E is a parallel sub-bundle, we also have that ν is a parallel field. Since Chen and Yano's have shown in [2] that a submanifold M contained in $\mathbb{R}^n$, $n \geqslant 4$, lies in a 4-sphere if and only if M admits a parallel umbilic normal field, we have the required result. We now observe that $\kappa_\nu(p) = \nu \cdot H_{E_p^{\perp *}} = d(p, \mathcal{A}ff_p) = \kappa_u(p)$, $\forall p \in M$. And since the ν-principal curvature κ_ν coincides with the curvature of the 4-sphere, the second assertion also follows. $\square$

In order to illustrate Theorem 5.5 we consider the Veronese surface $V \subset S^4$. This is defined as follows: consider the map

$$\begin{aligned} \xi : \mathbb{R}^3 &\longrightarrow \mathbb{R}^6, \\ (x, y, z) &\longmapsto \big(x^2, y^2, z^2, \sqrt{2}xy, \sqrt{2}xz, \sqrt{2}yz\big), \end{aligned}$$

the restriction of ξ to the 2-sphere $S = \{(x, y, z) \in \mathbb{R}^3;\ x^2 + y^2 + z^2 = 1\}$ gives rise to an embedding of the projective plane in $\mathbb{R}^6$, known as the Veronese surface. This surface is contained in a 4-sphere of a hyperplane in $\mathbb{R}^6$ and the curvature ellipse at each one of its points is a circle. Moreover, the mean curvature vector field $H(p)$ has constant norm and constant angle with E_p. Thus, $\kappa_u \equiv \sqrt{\frac{3}{2}}$ is a constant function, as expected.

We finally relate the behavior of the umbilical curvature with the problem of isometric reduction of the codimension. This problem has been considered for the general case of submanifolds of Riemannian manifolds with constant sectional curvature in [25]. A convenient manipulation of the conditions stated in the well known Fundamental Theorem for Submanifolds [26], leads to the following:

Proposition 5.6. *(See [25].) Let M be a simply connected n-submanifold of $\mathbb{R}^{n+k}$ and suppose that N^1M has constant rank $r < k$. Then there exists an isometric immersion ψ of M into $\mathbb{R}^{n+r}$ that extends to a vector bundle isomorphism between N^1M and the normal bundle of $\psi(M)$ in $\mathbb{R}^{n+r}$.*

We point out that the simply connectedness condition of the above proposition is required by the Fundamental Theorem for Submanifolds. So we can always ensure the existence of the isometric immersion in the local situation, but the global extension may fail along non-homotopicaly trivial paths.

We can interpret the above result in our context as follows:

Remark 5.7. Let M be a simply connected surface immersed in $\mathbb{R}^n$, $n \geqslant 5$, we have:

1. For $n \geqslant 6$, if the first normal bundle N^1M has constant rank $r = 3$, then there exists an isometric immersion ψ of M into $\mathbb{R}^5$ that extends to a vector bundle isomorphism between N^1M and the normal bundle of $\psi(M)$ in $\mathbb{R}^5$. This isomorphism takes the curvature ellipses of M in $\mathbb{R}^n$ to the curvature ellipses of $\psi(M)$ in $\mathbb{R}^5$.
2. For $n \geqslant 5$, if the first normal bundle N^1M has constant rank r, $r = 2, 1$, then there exists an isometric immersion ψ of M into $\mathbb{R}^{2+r}$ that extends to a vector bundle isomorphism between N^1M and the normal bundle of $\psi(M)$ in $\mathbb{R}^{r+2}$. This isomorphism takes the curvature ellipses of M in $\mathbb{R}^n$ to the curvature ellipses of $\psi(M)$ in $\mathbb{R}^{r+2}$.

In the following result we provide sufficient conditions in terms of the umbilical curvature that ensure the existence of isometric immersions of surfaces in $\mathbb{R}^n$, $n \geqslant 5$, into p-spheres and p-planes, $p = 3, 4, 5$. We point out that these isometric immersions are especial in the sense that preserve the "relevant" (non-vanishing) part of the second fundamental form, that is, they preserve the curvature ellipses and principal configurations of the considered surfaces as well as the first normal bundle.

Theorem 5.8.

1. *Let M be a simply connected surface immersed in $\mathbb{R}^n$, $n \geqslant 6$, with non-vanishing constant umbilical curvature, all whose points are non-semiumbilic. Then M admits an isometric immersion into a 5-plane.*
2. *Let M be a simply connected surface immersed in $\mathbb{R}^n$, $n \geqslant 5$, with vanishing constant umbilical curvature, all whose points are non-semiumbilic. Then M admits an isometric immersion into an affine 4-dimensional subspace.*
3. *Let M be a simply connected surface immersed in $\mathbb{R}^n$, $n \geqslant 5$, with non-vanishing constant umbilical curvature, all whose points are semiumbilic. Then M admits an isometric immersion into a 3-sphere.*
4. *Let M be a simply connected surface immersed in $\mathbb{R}^n$, $n \geqslant 5$, with vanishing constant umbilical curvature, all whose points are semiumbilic. Then M admits an isometric immersion into an affine 3-dimensional subspace.*

Proof. The first assertion follows from the fact that under the given conditions curvature ellipse non-degenerate at p and $\mathcal{A}ff_p$ does not pass through p, $\forall p \in M$. Therefore N^1M has constant rank 3.

On the other hand, the second assertion means that $\mathcal{A}ff_p$ is a plane through the origin at every point of the M and thus N^1M has constant rank 2 leading to the required result.

For the third, we observe that since κ_u is nonzero there are no inflection points and hence N^1M has constant rank 2 all over M. It then follows that there exists an isometric immersion ψ of M into $\mathbb{R}^4$ that extends to a vector bundle isomorphism between N^1M and the normal bundle of $\psi(M)$ in $\mathbb{R}^4$. Therefore $\psi(M)$ is a semiumbilical surface with constant umbilical curvature in $\mathbb{R}^4$. But this implies that the unit in the orthogonal direction to the curvature segment at every point of $\psi(M)$ is an umbilic normal field with constant curvature. Since in codimension 2 an umbilic field with constant curvature is parallel (see Chen and Yano [2]), we have that $\psi(M)$ admits a parallel umbilic normal field and thus lies in a 3-sphere. So we can conclude that M admits an isometric immersion into S^3.

Finally, assertion 4 implies that all the points of M are inflection (non-umbilic) points and thus N^1M has constant rank 1, from which we get that it must admit an isometric immersion into $\mathbb{R}^3$. □

So we get that, in each one of these cases, up to an appropriate isomorphism of N^1M, that restricts to an isometric immersion of M, we can immerse the surface in either a Euclidean subspace of lower dimension or a sphere. Nevertheless, we cannot ensure that the surface M lies substantially in such subspaces. For instance, we can construct a translation surface associated to two curves in $\mathbb{R}^5$. It is not difficult to verify that provided these two curves lie in orthogonal subspaces this surface is totally made of semiumbilic points. Consider the particular case of the curves $\alpha : (0, 2\pi) \to \mathbb{R}^5$, given by $\alpha(s) = (\cos s, \sin s, 0, 0, 0)$ and $\beta : (0, 2\pi) \to \mathbb{R}^5$, given by $\beta(t) = (0, 0, a\cos t, a\sin t, 1 - a^2)$, $0 < a < 1$, the associated translation surface is defined by the immersion

$$\begin{aligned} \phi : (0, 2\pi) \times (0, 2\pi) &\longrightarrow \mathbb{R}^5, \\ (s, t) &\longmapsto \alpha(s) + \beta(t) = \bigl(\cos s, \sin s, a\cos t, a\sin t, 1 - a^2\bigr). \end{aligned}$$

It can be seen that this surface has constant umbilical curvature is $\frac{a}{1+a^2}$. On the other hand, it is easy to verify that this surface does not lie in a hyperplane and hence it cannot lie in a 3-sphere. More generally, given a plane curve α with constant curvature κ_α and a curve β substantially immersed in a 3-space with constant curvature κ_β, by immersing them in orthogonal affine subspaces in $\mathbb{R}^5$, we can obtain a translation surface all whose points are semiumbilic and whose umbilical curvature is given by $\frac{\kappa_\alpha \cdot \kappa_\beta}{\sqrt{\kappa_\alpha^2 + \kappa_\beta^2}}$, and thus constant. Clearly, it is possible to choose β in such a way that this surface does not lie in a 3-sphere.

6. Final comments on special contacts

The results on the possible contacts of surfaces in 5-space with hyperplanes and hyperspheres obtained in the previous sections are summarized in Table 1. Here, we observe:

a) If p is of type M_3 there are not osculating hyperplanes of corank 2 at p and the unique umbilical focus determines a unique umbilical focal hypersphere.

b) If p is of type M_2 and non-semiumbilic, then there is no umbilical focal hypersphere at p and the umbilic center goes to infinity.

c) In the case of a non-radial semiumbilic point p the intersection of all the umbilical focal hyperspheres (including the osculating hyperplane of corank 2 as a limit case) determines a 3-sphere that has a special contact with M at p.

Table 1

Type of ellipse curvature	$\dim(N_p^1M) = \operatorname{rank}\alpha_\phi(p)$	Contact of corank 2 with hyperplanes	Umbilical focal hyperspheres
non-degenerate	3	none	a unique 4-sphere centered in the line $E_p^\perp$
non-degenerate	2 (non-semiumbilic)	a unique hyperplane given by $T_pM \oplus N_p^1M$	none
degenerate into a non-radial segment	1 semiumbilic		a 1-parameter family of 4-spheres centered in a line of $E_p^\perp$ (plane)
degenerate into a radial segment	1 radial semiumbilic	a 1-parameter family $\mathbb{H}_v^4 = T_pM \oplus v^\perp$,	none
degenerate into a point $\neq$ origin	1 umbilic (non-flat)	$v \in N_p^1M \subset N_pM$	a 2 parameter family of 4-spheres centered in a plane of $E_p^\perp = N_pM$
degenerate into the origin	0 flat umbilic	all tangent hyperplanes	none

d) If p is a radial semiumbilic or a non-flat umbilic, then the family of corank 2 osculating hyperplanes determines, as a common intersection, a 3-plane which also has a special contact with M at p. Moreover, in the case of a non-flat umbilic p the intersection of this 3-plane with the unique umbilical focal hypersphere determines a 2-sphere that also has a special contact with M at p.

e) For surfaces immersed in higher-dimensional Euclidean spaces we can generalize the results of the above table by considering the appropriate codimensions of E_p and N_p^1M in N_pM.

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